

# EXTENDED MULTIFRACTAL FORMALISM OF SOME NON-DOUBLING MEASURES

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**ABSTRACT.** In a previous work [7] we constructed measures on symbolic spaces which satisfy an extended multifractal formalism (in the sense that Olsen's functions  $b$  and  $B$  differ and that their Legendre transforms have the expected interpretation in terms of dimensions). These measures are composed with a Gray code and projected onto the unit interval so to get doubling measures. Then we were able to show that the projected measure has the same Olsen's functions as the one it comes from and that it also fulfills the extended multifractal formalism. Here we show that the use of a Gray code is not necessary to get these results, although dealing with non doubling measures.

**Key words:** Multifractal analysis, extended multifractal formalism, inhomogeneous multinomial measures, Hausdorff dimension, packing dimension.

## 1. INTRODUCTION

Ben Nasr, Bhouri, and Heurteaux [3] constructed a class of measures whose Olsen's  $b$  and  $B$  functions differ. They first consider a Markov measure on the symbolic space  $\{0, 1\}^{\mathbb{N}}$  and project it on  $[0, 1]$  by using the usual dyadic representation  $\gamma$  of numbers. The Markov rules are chosen so that the projected measure is doubling. Actually this is equivalent to the following construction.

Given two different numbers  $a, a' \in (0, 1)$  and an increasing sequence  $T_k$  of positive integers such that  $\lim_{k \rightarrow \infty} T_{k+1}/T_k = \infty$ , consider the measure  $\mu$  on the symbolic space so defined: for any  $w = w_1 \cdots w_n \in \{0, 1\}^n$ ,

$$\mu([w]) = \prod_{j=1}^n \left( p_j^{1-w_j} (1 - p_j)^{w_j} \right),$$

where  $[w]$  stands for the cylinder defined by  $w$ , and,  $p_j = a$  if  $T_{2k-1} \leq j < T_{2k}$ , and  $p_j = a'$  if  $T_{2k} \leq j < T_{2k+1}$  for some  $k$ . If we denote by  $g$  the Gray code (this is a map from the symbolic space into itself which allows to enumerate cylinders of the same size in such a way that one

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*Date:* December 2014.

2010 *Mathematics Subject Classification.* Primary 28A80; Secondary 28A78.

The research is supported by the National Natural Science Foundation of China, No. 11271223, No. 11431007 and No. 11201256.

passes from an element to the next one by flipping one digit only), then the measure  $\nu$  considered in [3] is the image of  $\mu$  under  $\gamma \circ g$ .

Since  $\nu$  is doubling, when concerning multifractal analysis, it inherits from  $\mu$  (see [3, 4, 7]).

On the other hand, Barral, Ben Nasr and Peyrière [1] showed that the projection under  $\gamma$  of a Bernoulli measure satisfies the multifractal formalism although it is not doubling. This is why we investigate whether it was necessary to use a Gray code to obtain a measure on  $[0, 1]$  satisfying an extended multifractal formalism (or refined multifractal formalism, according to Barral's terminology [2]). Indeed, we are going to show that projections under  $\gamma$  of inhomogeneous multinomial measures have this property.

Barral [2] proved that, given two convex functions fulfilling fairly general conditions, there exists a compactly supported (always supported on a Cantor set), positive, and finite Borel measure  $\rho$  on  $\mathbb{R}$  whose  $\tau_\rho$  and  $\underline{\tau}_\rho$  functions are just the two given functions. Also, the author mentioned that for the measure  $\rho$  possessing the weak doubling properties (see Inequality (4)), if  $\dim X_\rho(\alpha) = \underline{\tau}_\rho^*(\alpha)$  for all  $\alpha$  over its domain, then  $b_\rho = \underline{\tau}_\rho$ ; similarly if  $\text{Dim } X_\rho(\alpha) = \tau_\rho^*(\alpha)$  for all  $\alpha$  over its domain, then  $B_\rho = \tau_\rho$  (these notations will be reminded later whereas for the definition of domain, the reader is referred to [2]).

In contrast, for the measures we consider, generally the  $\underline{\tau}$  functions are not convex (see Theorem 2), and the condition  $\text{Dim } X(\alpha) = \tau^*(\alpha)$  does not always hold (see Theorem 4). Moreover, the measures we construct in this article have full support.

This article is organized as follows. In Section 2, we recall the basic notations, definitions, and the constructions of inhomogeneous multinomial measures. In Section 3, we present our main results. Then we prove the two results in Section 4 and Section 5 respectively.

## 2. RECOLLECTIONS: NOTATIONS AND DEFINITIONS

**2.1. The Olsen's measures and functions.** We work on a metric space  $(\mathbb{X}, d)$  possessing the Besicovitch property:

*There exists a constant  $C_B \in \mathbb{N}$  such that, given any bounded subset  $\{x_i\}_{i \in I} \subseteq \mathbb{X}$  and any collection  $\{B(x_i, r_i)\}_{i \in I}$  of balls in  $\mathbb{X}$ , one can extract from it  $C_B$  countable families  $\{\{B(x_{j,k}, r_{j,k})\}_{k \geq 1}\}_{1 \leq j \leq C_B}$  so that*

- $\bigcup_{j,k} B(x_{j,k}, r_{j,k}) \supseteq \{x_i\}_{i \in I}$ ,
- for any  $j$  and  $k \neq k'$ ,  $B(x_{j,k}, r_{j,k}) \cap B(x_{j,k'}, r_{j,k'}) = \emptyset$ .

It is known that Euclidean spaces and ultrametric spaces fulfill this condition.

Let  $\mu$  be a Borel probability measure on  $\mathbb{X}$ . Denote by  $S_\mu$  the support of the measure  $\mu$ . For any  $\alpha \in \mathbb{R}$ , we denote by  $X_\mu(\alpha)$  the level set of

points whose local Hölder exponents assume the value  $\alpha$ :

$$X_\mu(\alpha) = \left\{ x \in S_\mu : \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}$$

For  $q, t \in \mathbb{R}$  and  $\delta > 0$ , we shall use the measures and premeasures  $\overline{\mathcal{H}}_{\mu, \delta}^{q, t}$ ,  $\overline{\mathcal{H}}_\mu^{q, t}$ ,  $\mathcal{H}_\mu^{q, t}$  and  $\overline{\mathcal{P}}_{\mu, \delta}^{q, t}$ ,  $\overline{\mathcal{P}}_\mu^{q, t}$ ,  $\mathcal{P}_\mu^{q, t}$  introduced by Olsen [5] and whose definitions are recalled in [7].

The functions  $\mathcal{H}_\mu^{q, t}$ ,  $\mathcal{P}_\mu^{q, t}$  and  $\overline{\mathcal{P}}_\mu^{q, t}$  provide each subset  $E$  of  $\mathbb{X}$  with dimensional indices:

$$\begin{aligned} b_{\mu, E}(q) &= \sup\{s : \mathcal{H}_\mu^{q, s}(E) = \infty\} = \inf\{s : \mathcal{H}_\mu^{q, s}(E) = 0\}, \\ B_{\mu, E}(q) &= \sup\{s : \mathcal{P}_\mu^{q, s}(E) = \infty\} = \inf\{s : \mathcal{P}_\mu^{q, s}(E) = 0\}, \\ \tau_{\mu, E}(q) &= \sup\{s : \overline{\mathcal{P}}_\mu^{q, s}(E) = \infty\} = \inf\{s : \overline{\mathcal{P}}_\mu^{q, s}(E) = 0\}. \end{aligned}$$

One sees that the Olsen's function  $b_{\mu, E}(q)$  is a multifractal extension of the Hausdorff dimension  $\dim E$  whereas  $B_{\mu, E}(q)$  is a multifractal extension of the packing dimension  $\text{Dim } E$ . For simplicity, when the context is clear, we will write  $b_\mu = b_{\mu, S_\mu}$ ,  $B_\mu = B_{\mu, S_\mu}$ , and  $\tau_\mu = \tau_{\mu, S_\mu}$ . Some basic properties can be found in [5].

There is an alternate way to compute the function  $\tau_\mu$  according to [4, 6]. Fix  $\lambda < 1$ , one has

$$(1) \quad \tau_\mu(q) = \limsup_{\delta \rightarrow 0} \frac{-1}{\log \delta} \log \sup \left\{ \sum_i \mu(B_i)^q : \right. \\ \left. (B_i)_i \text{ is a packing of } S_\mu \text{ with } \lambda\delta < r_i \leq \delta \right\},$$

where  $r_i$  is the radius of the ball  $B_i$ .

## 2.2. Inhomogeneous multinomial measures.

2.2.1. *The mixed symbolic spaces.* Let  $c_1, c_2 \geq 2$  be two positive integers,  $\mathcal{A}_1 = \{0, 1, \dots, c_1 - 1\}$ ,  $\mathcal{A}_2 = \{0, 1, \dots, c_2 - 1\}$  be two alphabets. Fix in this article a sequence of integers  $(T_k)$  such that

$$T_1 = 1, T_k < T_{k+1} \text{ and } \lim_{k \rightarrow \infty} T_{k+1}/T_k = +\infty.$$

Consider the set of infinite words

$$\partial \mathcal{A}_{1,2}^* = \mathcal{A}_1^{T_2 - T_1} \times \mathcal{A}_2^{T_3 - T_2} \times \mathcal{A}_1^{T_4 - T_3} \times \dots = \prod_j Y_j,$$

where

- if  $T_{2k-1} \leq j < T_{2k}$  for some  $k$ ,  $Y_j = \mathcal{A}_1$ ,
- if  $T_{2k} \leq j < T_{2k+1}$  for some  $k$ ,  $Y_j = \mathcal{A}_2$ .

We call  $\partial\mathcal{A}_{1,2}^*$  the mixed symbolic space with respect to the triplet  $\{\mathcal{A}_1, \mathcal{A}_2, (T_k)\}$ . Also we denote by  $\mathcal{A}_{1,2}^n$  the set of words of length  $n$  (by convention the empty word  $\epsilon$  has length 0), and  $\mathcal{A}_{1,2}^*$  the set of finite words, i.e.

$$\begin{aligned}\mathcal{A}_{1,2}^0 &= \{\epsilon\}, \\ \mathcal{A}_{1,2}^n &= \prod_{j=1}^n Y_j, \quad n \geq 1, \\ \mathcal{A}_{1,2}^* &= \bigcup_{n \geq 0} \mathcal{A}_{1,2}^n.\end{aligned}$$

The length of a finite word  $w$  is denoted by  $\ell(w)$ . If  $w = \varepsilon_1 \cdots \varepsilon_k \cdots$  is an infinite word or a finite word of length larger than  $k$ , the  $k$ -prefix of  $w$  is denoted by  $w|_k = \varepsilon_1 \cdots \varepsilon_k$ . And for any word  $w$ , by convention one has  $w|_0 = \epsilon$ , where  $\epsilon$  is the empty word. If  $w$  and  $v$  are two words,  $w \wedge v$  stands for their largest common prefix.

Let  $N_n$  be the number of integers  $j \leq n$  such that  $Y_j = \mathcal{A}_1$ . We can immediately get that

$$(2) \quad \liminf_{n \rightarrow \infty} \frac{N_n}{n} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{N_n}{n} = 1.$$

For any two different elements  $w, v \in \partial\mathcal{A}_{1,2}^*$  with  $\ell(w \wedge v) = n$ , we define  $d(w, v) = c_1^{-N_n} c_2^{-(n-N_n)}$ . It is easy to check that this defines an ultrametric distance on  $\partial\mathcal{A}_{1,2}^*$ .

Each finite word  $w \in \mathcal{A}_{1,2}^*$  defines a cylinder  $[w] = \{x \in \partial\mathcal{A}_{1,2}^* : x|_{\ell(w)} = w\}$ , which can also be viewed as a ball; and the diameter of this ball is denoted by  $|w|$ . For a Borel measure  $\mu$  on  $\partial\mathcal{A}_{1,2}^*$ , we simply write  $\mu([w]) = \mu(w)$ . Thus we identify the Borel measure  $\mu$  on  $\partial\mathcal{A}_{1,2}^*$  with a mapping from  $\mathcal{A}_{1,2}^*$  to  $[0, +\infty]$  subject to the following compatibility condition

$$\mu(w) = \sum_{\substack{x \in \mathcal{A}_{1,2}^{n+1} \\ x|_n = w}} \mu(x), \quad \text{for any } n \geq 0 \text{ and } w \in \mathcal{A}_{1,2}^n.$$

One sees that when the alphabets  $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ , the mixed symbolic space  $\partial\mathcal{A}_{1,2}^*$  becomes ordinary symbolic space  $\partial\mathcal{A}^*$ .

Since the radii of balls are discrete on the mixed symbolic space, the computation of  $\tau_\mu$  according to Formula (1) is quite easy. Indeed, for any  $n$ , we take any element  $w \in \mathcal{A}_{1,2}^n$  and define  $\tau_{\mu,n}$  by the following formula

$$(3) \quad \sum_{z \in \mathcal{A}_{1,2}^n} \mu(z)^q = |w|^{-\tau_{\mu,n}(q)}.$$

Then

$$\tau_\mu(q) = \limsup_{n \rightarrow \infty} \tau_{\mu,n}(q).$$

Also, we denote

$$\underline{\tau}_\mu(q) = \liminf_{n \rightarrow \infty} \tau_{\mu,n}(q).$$

**2.2.2. Image measures.** There is a natural map from  $\partial\mathcal{A}_{1,2}^*$  onto  $\mathbb{R}$ . Consider the map  $\gamma$  which sends the element  $x = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n \cdots$  to the real number  $\sum_{n \geq 1} \varepsilon_n c_1^{-N_n} c_2^{-(n-N_n)}$ . This map sends  $n$ -cylinders to basic intervals of  $n$ -th generation and thus defines a function on  $\mathcal{A}_{1,2}^*$ , still denoted by  $\gamma$ . To be precise, one can assign to each  $w \in \mathcal{A}_{1,2}^n$  an integer  $\iota(w)$  such that

$$\gamma(w) = \left[ \iota(w) c_1^{-N_n} c_2^{-(n-N_n)}, (\iota(w) + 1) c_1^{-N_n} c_2^{-(n-N_n)} \right].$$

Now if  $\mu$  is a Borel probability measure on  $(\partial\mathcal{A}_{1,2}^*, d)$ , we have its projection  $\nu$  on  $(\mathbb{R}, |\cdot|)$ . This is the image measure  $\nu = \gamma_*(\mu)$ , defined by  $\nu(E) = \mu(\gamma^{-1}(E))$  for any Borel set  $E \subseteq [0, 1]$ .

In particular case, the projection under  $\gamma$  of a Bernoulli measure satisfies the multifractal formalism, although it is not doubling. In fact, J. Barral, F. Ben Nasr and J. Peyrière proved the following stronger result.

**Theorem 1** (see [1]). *Let  $\mu$  be a continuous quasi-Bernoulli measure on  $\partial\mathcal{A}^*$ . Then both measures  $\mu$  and  $\nu = \gamma_*(\mu)$  obey the multifractal formalism everywhere and one has*

$$b_\nu = B_\nu = \tau_\nu = b_\mu = B_\mu = \tau_\mu.$$

However in general, the measure we are going to study is not quasi-Bernoulli.

**2.2.3. Inhomogeneous multinomial measures.** Given two groups of real numbers  $a_i, b_j \in (0, 1) (i = 1, \dots, c_1, j = 1, \dots, c_2)$  satisfying

$$a_1 + \cdots + a_{c_1} = b_1 + \cdots + b_{c_2} = 1,$$

we define a probability measure  $\mu$  on  $\partial\mathcal{A}_{1,2}^*$  that we call an inhomogeneous multinomial measure as explained below. As in [7], for every cylinder  $[\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n]$ , we set

$$\mu(\varepsilon_1 \cdots \varepsilon_n) = \prod_{j=1}^n p_j,$$

where

- if  $T_{2k-1} \leq j < T_{2k}$  for some  $k$ ,  $p_j = a_{\varepsilon_j+1}$ ,
- if  $T_{2k} \leq j < T_{2k+1}$  for some  $k$ ,  $p_j = b_{\varepsilon_j+1}$ .

Then we compute the  $\tau_\mu$  function. As previously, let  $N_n$  stand for the number of integers  $j \leq n$  such that  $p_j \in \{a_1, \dots, a_{c_1}\}$ , then we obtain by Formula (3) that

$$\tau_{\mu,n}(q) = \frac{\frac{N_n}{n} \log(a_1^q + \dots + a_{c_1}^q)}{\frac{N_n}{n} \log c_1 + (1 - \frac{N_n}{n}) \log c_2} + \frac{(1 - \frac{N_n}{n}) \log(b_1^q + \dots + b_{c_2}^q)}{\frac{N_n}{n} \log c_1 + (1 - \frac{N_n}{n}) \log c_2}.$$

This combined with (2) implies

$$\tau_\mu(q) = \max\{\log_{c_1}(a_1^q + \dots + a_{c_1}^q), \log_{c_2}(b_1^q + \dots + b_{c_2}^q)\},$$

$$\underline{\tau}_\mu(q) = \min\{\log_{c_1}(a_1^q + \dots + a_{c_1}^q), \log_{c_2}(b_1^q + \dots + b_{c_2}^q)\}.$$

Recall the following fact.

**Theorem 2** (see [7]). *One has*

$$B_\mu(q) = \tau_\mu(q) = \max\{\log_{c_1}(a_1^q + \dots + a_{c_1}^q), \log_{c_2}(b_1^q + \dots + b_{c_2}^q)\},$$

$$b_\mu(q) = \underline{\tau}_\mu(q) = \min\{\log_{c_1}(a_1^q + \dots + a_{c_1}^q), \log_{c_2}(b_1^q + \dots + b_{c_2}^q)\}.$$

We need the following auxiliary measures.

**Lemma 3** (see [3, 7]). *For any  $q \in \mathbb{R}$ , there is a probability measure  $\mu_q$  on  $\partial \mathcal{A}_{1,2}^*$  and a subsequence of integers  $(n_k)_{k \geq 1}$ , such that*

$$\mu_q(w) = \mu(w)^q |w|^{\tau_{\mu,n}(q)}, \text{ if } w \in \mathcal{A}_{1,2}^n.$$

Moreover,

$$\mu_q(w) \leq \mu(w)^q |w|^{\underline{\tau}_\mu(q)}, \text{ if } w \in \mathcal{A}_{1,2}^n,$$

and for every  $\varepsilon > 0$ ,

$$\mu_q(w) \leq \mu(w)^q |w|^{\tau_\mu(q) - \varepsilon}, \text{ if } w \in \mathcal{A}_{1,2}^{n_k} \text{ with } k \text{ large.}$$

We also point out that all these measures  $\mu, \mu_q$  are continuous.

### 3. MAIN RESULTS

Let us state our main results. Denote

$$\theta_a(q) = \log_{c_1}(a_1^q + \dots + a_{c_1}^q),$$

$$\theta_b(q) = \log_{c_2}(b_1^q + \dots + b_{c_2}^q).$$

And

$$s_1 = \min \left\{ \max_{1 \leq i \leq c_1} \{a_i\}, \max_{1 \leq j \leq c_2} \{b_j\} \right\},$$

$$s_2 = \max \left\{ \min_{1 \leq i \leq c_1} \{a_i\}, \min_{1 \leq j \leq c_2} \{b_j\} \right\}.$$

Let  $f^*(x) = \inf_y(xy + f(y))$  denote the Legendre transform of the function  $f$ .

**Theorem 4.** Let  $\mathcal{A}_1 = \{0, 1, \dots, c_1 - 1\}$ ,  $\mathcal{A}_2 = \{0, 1, \dots, c_2 - 1\}$  and let  $\mu$  be the probability measure on  $\partial\mathcal{A}_{1,2}^*$  taken from Theorem 2. Denote  $\nu = \gamma_*(\mu)$ . Then for every  $q \in \mathbb{R}$ ,

$$B_\nu(q) = B_\mu(q) = \max\{\theta_a(q), \theta_b(q)\},$$

$$b_\nu(q) = b_\mu(q) = \min\{\theta_a(q), \theta_b(q)\}.$$

**Theorem 5.** For any  $\alpha \in (-\log s_1, -\log s_2)$ , we have

$$\dim X_\mu(\alpha) = \dim X_\nu(\alpha) = b_\mu^*(\alpha) = b_\nu^*(\alpha).$$

And for  $\alpha \in (-\log s_1, -\log s_2)$  subject to

$$\max\{\theta_a^*(\alpha), \theta_b^*(\alpha)\} = B_\mu^*(\alpha),$$

we have

$$\text{Dim } X_\mu(\alpha) = \text{Dim } X_\nu(\alpha) = B_\mu^*(\alpha) = B_\nu^*(\alpha).$$

#### 4. PROOF OF THEOREM 4

To avoid tedious notations, we write the proof with  $c_1 = c_2 = 2$ . The reader will realize that the general case can be handled with minor modifications. For  $n \geq 0$ , we denote by  $\mathcal{F}_n$  the family of basic intervals of  $n$ -th generation:

$$\mathcal{F}_n = \left\{ \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right], 0 \leq j \leq 2^n - 1 \right\}.$$

For  $x \in (0, 1)$ , denote by  $I_n(x)$  the basic interval of  $n$ -th generation which contains  $x$  when  $x$  is not of the form  $k2^{-m}$ . When  $x$  is dyadic there are two basic intervals of  $n$ -th generation containing  $x$  and we choose the left one to be  $I_n(x)$ . We also denote by  $I_n(x)^\pm$  the basic intervals of the same generation, which are the right and left neighbors of  $I_n(x)$  respectively.

The following key lemma says that the measure  $\nu = \gamma_*(\mu)$  exhibits some weak doubling behaviors.

**Lemma 6.** Let  $\varpi$  be a continuous probability measure on  $[0, 1]$  satisfying that, there exists a positive constant  $C_1 < 1$ , such that for any  $n \geq 0$ , for any  $J \in \mathcal{F}_n$ , and for  $I \in \mathcal{F}_{n+1}$  with  $I \subseteq J$ , one has  $\varpi(I) \leq C_1 \varpi(J)$ .

Then there exists a constant  $C_0 > 1$  such that for  $\varpi$ -almost every  $x \in [0, 1]$ , when  $n$  is large enough,

$$(4) \quad C_0^{-(\sqrt{n}+1)} \leq \frac{\nu(I_n(x))}{\nu(I_n(x)^\pm)} \leq C_0^{\sqrt{n}+1}.$$

*Proof.* Denote  $\langle n \rangle = \lfloor n - \sqrt{n} \rfloor$ . We define the set

$$E_n = \{x \in [0, 1] : \ell(\gamma^{-1}(I_n(x)) \wedge \gamma^{-1}(I_n(x)^+)) < \langle n \rangle \text{ or} \\ \ell(\gamma^{-1}(I_n(x)) \wedge \gamma^{-1}(I_n(x)^-)) < \langle n \rangle\}.$$

We fix  $n$  for the moment and denote the basic intervals of  $\langle n \rangle$ -th generation, from left to right, by  $I_1, \dots, I_{k_n}$ , where  $k_n = 2^{\langle n \rangle}$ . The leftmost and rightmost basic subinterval, of  $n$ -th generation, of  $I_j$  are denoted by  $I_j^{(1)}$  and  $I_j^{(2)}$  respectively. So by definition

$$E_n \subseteq \bigcup_{j=1}^{k_n} I_j^{(1)} \cup I_j^{(2)}.$$

As an example, we compare the  $\varpi$ -mass of  $I_j^{(1)}$  with its mother interval, denoted by  $J$ . Note that  $I_j^{(1)} \in \mathcal{F}_n$  and  $J \in \mathcal{F}_{n-1}$ . Thus it follows that  $\varpi(I_j^{(1)}) \leq C_1 \varpi(J)$ , which implies  $\varpi(I_j^{(1)}) \leq C_1^{\sqrt{n}} \varpi(I_j)$ . Moreover,

$$\varpi(E_n) \leq \sum_{j=1}^{k_n} (\varpi(I_j^{(1)}) + \varpi(I_j^{(2)})) \leq 2C_1^{\sqrt{n}} \sum_{j=1}^{k_n} \varpi(I_j) = 2C_1^{\sqrt{n}},$$

from which it follows

$$\sum_{n \geq 1} \varpi(E_n) \leq \sum_{n \geq 1} 2C_1^{\sqrt{n}} < +\infty.$$

By Borel-Cantelli lemma, one immediately gets

$$\varpi(\limsup_n E_n) = 0,$$

which implies

$$\varpi(\liminf_n E_n^c) = 1,$$

where  $E_n^c = [0, 1] \setminus E_n$ .

To conclude, one recalls that  $\ell(\gamma^{-1}(I_n(x)) \wedge \gamma^{-1}(I_n(x)^\pm)) \geq \langle n \rangle$  if  $x \in E_n^c$ , meaning that the  $\nu$ -masses will not differ too much. In fact, denote

$$C_0 = \max \left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1}, \frac{b_1}{b_2}, \frac{b_2}{b_1} \right\}.$$

Then  $C_0 > 1$  and

$$C_0^{-(\sqrt{n}+1)} \leq \frac{\nu(I_n(x))}{\nu(I_n(x)^\pm)} \leq C_0^{\sqrt{n}+1}.$$

□

**Corollary 7.** *Recall the auxiliary measures  $\mu_q$  presented in Lemma 3. For any  $q \in \mathbb{R}$ , let  $\nu_q = \gamma_*(\mu_q)$ , then for  $\nu_q$ -almost every  $x \in [0, 1]$ , when  $n$  is large enough, Inequality (4) holds.*

*Proof.* For any  $q$ , let

$$C_1(q) = \max \left\{ \frac{a_1^q}{a_1^q + a_2^q}, \frac{a_2^q}{a_1^q + a_2^q}, \frac{b_1^q}{b_1^q + b_2^q}, \frac{b_2^q}{b_1^q + b_2^q} \right\}.$$

Then  $C_1(q) < 1$  and for any  $J \in \mathcal{F}_n$ , for  $I \in \mathcal{F}_{n+1}$  with  $I \subseteq J$ , one obtains by Lemma 3 that  $\nu_q(I) \leq C_1(q) \nu_q(J)$ . □



4.1.  $b_\nu(q) = \underline{\tau}_\mu(q)$ . We first consider the easier part  $b_\nu(q) \leq \underline{\tau}_\mu(q)$ . For any  $\varepsilon > 0$ , choose a subsequence  $\{n_k\}$  such that  $\tau_{\mu, n_k}(q) < \underline{\tau}_\mu(q) + \varepsilon$ , for every  $k \geq 1$ . Take any subset  $F \subseteq S_\nu = [0, 1]$ , and we choose the natural centered  $2^{-n_k}$ -covering of  $F$ , which is a set of all basic intervals of  $n_k$ -th generation. Now

$$\overline{\mathcal{H}}_{\nu, 2^{-n_k}}^{q, \underline{\tau}_\mu(q) + \varepsilon}(F) \leq \sum_{I \in \mathcal{F}_{n_k}} \nu(I)^q 2^{-n_k(\underline{\tau}_\mu(q) + \varepsilon)} = 2^{n_k(\tau_{\mu, n_k}(q) - \underline{\tau}_\mu(q) - \varepsilon)} \leq 1,$$

which implies

$$\overline{\mathcal{H}}_\nu^{q, \underline{\tau}_\mu(q) + \varepsilon}(F) \leq 1,$$

and

$$\mathcal{H}_\nu^{q, \underline{\tau}_\mu(q) + \varepsilon}(S_\nu) \leq 1.$$

Thus  $b_\nu(q) \leq \underline{\tau}_\mu(q)$  since  $\varepsilon$  is arbitrary.

Next we turn to the opposite part  $b_\nu(q) \geq \underline{\tau}_\mu(q)$ .

**Lemma 8.** *Let  $q \in \mathbb{R}$ . For any  $\varepsilon > 0$ , and for  $\nu_q$ -almost every  $x$ , when  $r$  is small enough,*

$$\nu_q(B(x, r)) \leq \nu(B(x, r))^q r^{\underline{\tau}_\mu(q) - \varepsilon}.$$

*Then we have  $b_\nu(q) \geq \underline{\tau}_\mu(q)$ .*

*Proof.* For  $\nu_q$ -almost every  $x \in [0, 1]$ , when  $r$  is small enough, we can choose  $n$  large enough such that  $2^{-(n+2)} < r \leq 2^{-(n+1)}$ . So there exist at most two basic intervals  $I_1, I_2 \in \mathcal{F}_n$  such that  $B(x, r) \subseteq I_1 \cup I_2$ ; on the other hand,  $B(x, r)$  must contain at least one basic interval of  $(n+2)$ -th generation, denoted by  $I_3$ . It is no restriction to assume that  $x \in I_1$  and  $x \in I_3$ . Note that by Corollary 7,  $\nu(I_1)$  and  $\nu(I_2)$  do not differ too much since  $n$  is large. With the help of Lemma 3, we have

$$\begin{aligned} \nu_q(B(x, r)) &\leq \nu_q(I_1) + \nu_q(I_2) \leq \nu(I_1)^q |I_1|^{\underline{\tau}_\mu(q)} + \nu(I_2)^q |I_2|^{\underline{\tau}_\mu(q)} \\ &= (\nu(I_1)^q + \nu(I_2)^q) |I_1|^{\underline{\tau}_\mu(q)}, \end{aligned}$$

where  $|I|$  stands for the length of the interval  $I$ .

When  $q < 0$ , we first consider the case where  $\nu(I_1) \geq \frac{1}{2}\nu(B(x, r))$ . We then have

$$\nu(I_2) \geq \frac{1}{C_0^{\sqrt{n}+1}} \nu(I_1) \geq \frac{1}{2C_0^{\sqrt{n}+1}} \nu(B(x, r)),$$

which implies

$$\begin{aligned} \nu_q(B(x, r)) &\leq \left( \frac{1}{2^q} + \frac{1}{2^q C_0^{q(\sqrt{n}+1)}} \right) \nu(B(x, r))^q |I_1|^{\underline{\tau}_\mu(q)} \\ &\leq C(q) C_0^{-q\sqrt{n}} \nu(B(x, r))^q r^{\underline{\tau}_\mu(q)} \\ &= \nu(B(x, r))^q r^{\underline{\tau}_\mu(q) - \varsigma(q, r)}, \end{aligned}$$

where

$$C(q) = 2^{-(q-1)} C_0^{-q} \max\{2^{\underline{\tau}_\mu(q)}, 4^{\underline{\tau}_\mu(q)}\},$$

and

$$\varsigma(q, r) = \frac{\log C(q) - q\sqrt{n} \log C_0}{-\log r}.$$

So for any  $\varepsilon > 0$ , take  $r$  small enough such that  $\varsigma(q, r) < \varepsilon$ , then we have

$$\nu_q(B(x, r)) \leq \nu(B(x, r))^q r^{\mathcal{I}_\mu(q) - \varepsilon}.$$

With the same method, in the case where  $\nu(I_2) \geq \frac{1}{2}\nu(B(x, r))$ , we can get the same results.

When  $q \geq 0$ , we use the fact that  $\nu(B(x, r)) \geq \nu(I_3)$ . Since  $I_1$  is the grandmother interval of  $I_3$ , we have  $\nu(I_3) \geq C_2^2 \nu(I_1)$  (where  $C_2 = \min\{a_1, a_2, b_1, b_2\} < 1$ ), which implies

$$\nu(I_1) \leq C_2^{-2} \nu(I_3) \leq C_2^{-2} \nu(B(x, r)),$$

and thus

$$\nu(I_2) \leq C_0^{\sqrt{n}+1} \nu(I_1) \leq C_0^{\sqrt{n}+1} C_2^{-2} \nu(B(x, r)).$$

Then in the very same way as above, we also conclude that for any  $\varepsilon > 0$ , when  $r$  is small enough, we have

$$\nu_q(B(x, r)) \leq \nu(B(x, r))^q r^{\mathcal{I}_\mu(q) - \varepsilon}.$$

Finally, we have to show  $b_\nu(q) \geq \mathcal{I}_\mu(q)$ . For any  $\varepsilon > 0$ , denote

$$W = \{x \in [0, 1] : \exists r_x > 0, \forall r < r_x, \nu_q(B(x, r)) \leq \nu(B(x, r))^q r^{\mathcal{I}_\mu(q) - \varepsilon}\},$$

$$W_n = \{x \in W : \forall r < 1/n, \nu_q(B(x, r)) \leq \nu(B(x, r))^q r^{\mathcal{I}_\mu(q) - \varepsilon}\}, \quad \forall n \in \mathbb{N}.$$

Then  $\nu_q(W) = 1$  and  $W_n \uparrow W$ . Take  $n$  with  $\nu_q^*(W_n) > 0$ , where  $\nu_q^*$  stands for the outer measure of  $\nu_q$ .

For any centered  $\frac{1}{n}$ -covering  $\{B_j\}$  of  $W_n$ ,

$$0 < \nu_q^*(W_n) \leq \sum \nu_q^*(B_j) \leq \sum \nu_q(B_j) \leq \sum \nu(B_j)^q r_j^{\mathcal{I}_\mu(q) - \varepsilon},$$

which implies

$$\overline{\mathcal{H}}_{\nu, \frac{1}{n}}^{q, \mathcal{I}_\mu(q) - \varepsilon}(W_n) > 0,$$

and

$$\mathcal{H}_\nu^{q, \mathcal{I}_\mu(q) - \varepsilon}(S_\nu) > 0.$$

By definition this means  $b_\nu(q) \geq \mathcal{I}_\mu(q) - \varepsilon$ . Since  $\varepsilon$  is arbitrary, we conclude that  $b_\nu(q) \geq \mathcal{I}_\mu(q)$ .  $\square$

4.2.  $B_\nu(q) = \tau_\mu(q)$ . The following lemma gives a general estimate.

**Lemma 9** (see [1]). *Let  $\mu$  be a probability measure on  $\partial\mathcal{A}^*$  and let  $\nu = \gamma_*(\mu)$  be its projection onto  $[0, 1]$ . One has  $\tau_\nu(q) \leq \tau_\mu(q)$ .*

So it is sufficient to show that  $B_\nu(q) \geq \tau_\mu(q)$ . Following the spirit of Lemma 8, we introduce

**Lemma 10.** *Let  $q \in \mathbb{R}$ . For any  $\varepsilon > 0$ , for  $\nu_q$ -almost every  $x \in [0, 1]$ , for any  $\delta > 0$ , there exists  $r < \delta$ , such that*

$$\nu_q(B(x, r)) \leq \nu(B(x, r))^{q\tau_\mu(q)-\varepsilon}.$$

Then we have  $B_\nu(q) \geq \tau_\mu(q)$ .

*Proof.* The proof of the first assertion follows the same lines as the proof of the first assertion of Lemma 8. So it is sufficient to only prove the second. One takes any family of  $\{E_i\}$  such that  $\cup E_i = S_\nu = [0, 1]$  and for each  $i$  one computes  $\overline{\mathcal{P}}_\nu^{q, \tau_\mu(q)-\varepsilon}(E_i)$ .

Denote by  $N$  the  $\nu_q$ -null set, i.e. for any  $x \in [0, 1] \setminus N$ , for any  $\delta > 0$ , there exists  $r < \delta$ , such that  $\nu_q(B(x, r)) \leq \nu(B(x, r))^{q\tau_\mu(q)-\varepsilon}$ .

By Besicovitch property, we can extract from  $\{B(x, r)\}_{x \in E_i \setminus N}$   $C_B$  countable families  $\{B_{j,k}\}_{1 \leq j \leq C_B, k \geq 1}$  such that  $\cup_{j,k} B_{j,k} \supseteq E_i \setminus N$  and for any  $j$ ,  $\{B_{j,k}\}_{k \geq 1}$  is a  $\delta$ -packing of  $E_i \setminus N$ .

Then one gets

$$\nu_q^*(E_i \setminus N) \leq \sum_{j,k} \nu_q^*(B_{j,k}) \leq \sum_{j,k} \nu_q(B_{j,k}) \leq \sum_{j,k} \nu(B_{j,k})^{q\tau_\mu(q)-\varepsilon}.$$

So there exists  $j$  such that  $\sum_k \nu(B_{j,k})^{q\tau_\mu(q)-\varepsilon} \geq \frac{1}{C_B} \nu_q^*(E_i \setminus N)$ . Thus

$$\overline{\mathcal{P}}_\nu^{q, \tau_\mu(q)-\varepsilon}(E_i \setminus N) \geq \frac{1}{C_B} \nu_q^*(E_i \setminus N),$$

which implies

$$\overline{\mathcal{P}}_\nu^{q, \tau_\mu(q)-\varepsilon}(E_i) \geq \frac{1}{C_B} \nu_q^*(E_i),$$

and

$$\sum_i \overline{\mathcal{P}}_\nu^{q, \tau_\mu(q)-\varepsilon}(E_i) \geq \frac{1}{C_B} \sum_i \nu_q^*(E_i) \geq \frac{1}{C_B} \nu_q^*(S_\nu),$$

yielding that

$$\mathcal{P}_\nu^{q, \tau_\mu(q)-\varepsilon}(S_\nu) \geq \frac{1}{C_B} \nu_q^*(S_\nu) > 0.$$

Again by definition this means  $B_\nu(q) \geq \tau_\mu(q) - \varepsilon$ . Since  $\varepsilon$  is arbitrary,  $B_\nu(q) \geq \tau_\mu(q)$ .  $\square$

So together with Lemma 9, we conclude that  $B_\nu(q) = \tau_\mu(q)$ . And the proof of Theorem 4 is complete.

## 5. THE PROOF OF THEOREM 5

We already know in [4, 7] that for a certain range of  $\alpha$ , the Hausdorff dimension of the set  $X_\mu(\alpha)$  is given by the value of the Legendre transform of  $b_\mu$  at  $\alpha$  whereas its packing dimension is the value of the Legendre transform of  $B_\mu$  at  $\alpha$ . This means that the measure  $\mu$  satisfies an extended multifractal formalism at  $\alpha$ . In this section, we show that for the same range of  $\alpha$ , the image measure  $\nu$  also has this property. So the use of a Gray code is not necessary although we are dealing with non-doubling measures. As previously, we still set  $c_1 = c_2 = 2$ .

For  $\alpha \in (-\log s_1, -\log s_2)$ , one finds  $q_a, q_b$  such that

$$-\theta'_a(q_a) = -\theta'_b(q_b) = \alpha.$$

Then one defines a new probability measure  $\tilde{\mu}_q$  on the symbolic space just as  $\mu$  by replacing  $\{a_1, a_2, b_1, b_2\}$  with  $\{\tilde{a}_1, \tilde{a}_2, \tilde{b}_1, \tilde{b}_2\}$ , where

$$(5) \quad \tilde{a}_1 = \frac{a_1^{q_a}}{a_1^{q_a} + a_2^{q_a}}, \quad \tilde{b}_1 = \frac{b_1^{q_b}}{b_1^{q_b} + b_2^{q_b}}.$$

And  $\tilde{a}_2 = 1 - \tilde{a}_1$ ,  $\tilde{b}_2 = 1 - \tilde{b}_1$ . We also denote  $\tilde{\nu}_q = \gamma_*(\tilde{\mu}_q)$ .

Fix  $\lambda < 1$  and define

$$\varphi(t) = \limsup_{\delta \rightarrow 0} \frac{-1}{\log \delta} \log \sup \left\{ \sum_i \mu(B_i)^t \tilde{\mu}_q(B_i) : \right. \\ \left. (B_i)_i \text{ packing of } S_\mu \text{ with } \lambda\delta < r_i \leq \delta \right\},$$

then it is easy to compute

$$\varphi(t) = \log_2 \max\{a_1^t \tilde{a}_1 + a_2^t \tilde{a}_2, b_1^t \tilde{b}_1 + b_2^t \tilde{b}_2\}.$$

So  $\varphi(0) = 0$ . And the method of choosing  $\{\tilde{a}_1, \tilde{b}_1\}$  in Formula (5) insures that  $\varphi'(0)$  exists. In fact,

$$-\varphi'(0) = -\tilde{a}_1 \log_2 a_1 - \tilde{a}_2 \log_2 a_2 = -\tilde{b}_1 \log_2 b_1 - \tilde{b}_2 \log_2 b_2 = \alpha.$$

Thus by [4, 6] the set  $X_\mu(\alpha)$  has full  $\tilde{\mu}_q$ -mass, and this implies that the set  $X_\nu(\alpha)$  has full  $\tilde{\nu}_q$ -mass. Actually we have

**Lemma 11.** *For  $\tilde{\nu}_q$ -almost every  $x \in [0, 1]$ ,*

$$\liminf_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r} = \liminf_{n \rightarrow \infty} \frac{\log \nu(I_n(x))}{\log 2^{-n}}, \\ \limsup_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r} = \limsup_{n \rightarrow \infty} \frac{\log \nu(I_n(x))}{\log 2^{-n}}.$$

*Proof.* First we revisit Lemma 6. We can obtain a constant  $C_0 > 1$  and a subset  $N \subseteq [0, 1]$  such that  $\tilde{\nu}_q(N) = 0$ , and that for any  $x \in [0, 1] \setminus N$ , when  $n$  is large enough,

$$(6) \quad C_0^{-(\sqrt{n}+1)} \leq \frac{\nu(I_n(x))}{\nu(I_n(x)^\pm)} \leq C_0^{\sqrt{n}+1}.$$

For any  $x \in [0, 1] \setminus N$ , when  $r$  is small enough, we can choose  $n$  large enough such that  $2^{-(n+2)} < r \leq 2^{-(n+1)}$ . So there exist at most two basic intervals  $I_1, I_2 \in \mathcal{F}_n$  such that  $B(x, r) \subseteq I_1 \cup I_2$ ; on the other hand,  $B(x, r)$  must contain at least one basic interval of  $(n+2)$ -th generation, denoted by  $I_3$ . It is no restriction to assume that  $x \in I_1$  and  $x \in I_3$ .

Since  $n$  is large, by (6) we have

$$\nu(B(x, r)) \leq \nu(I_1) + \nu(I_2) \leq (1 + C_0^{\sqrt{n}+1})\nu(I_1).$$

And since  $I_3 \subseteq I_1$ ,

$$\nu(B(x, r)) \geq \nu(I_3) \geq C'_0 \nu(I_1),$$

where  $C'_0 = \min\{a_i a_j, a_i b_j, b_i b_j : i, j = 1, 2\}$ .

To conclude, one notices that  $I_1 = I_n(x)$ , and obtains

$$\liminf_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r} = \liminf_{n \rightarrow \infty} \frac{\log \nu(I_n(x))}{\log 2^{-n}}.$$

And the same goes for the upper limit.  $\square$

So we obtain from this lemma that  $X_\nu(\alpha) \setminus N = \gamma(X_\mu(\alpha)) \setminus N$ , while  $\tilde{\nu}_q(N) = 0$ . Thus  $\tilde{\nu}_q(X_\nu(\alpha)) = 1$ , which yields

**Lemma 12** (see [4]). *Let  $E = X_\nu(\alpha)$ , then one has*

$$\dim E \geq \operatorname{ess\,sup}_{x \in E, \tilde{\nu}_q} \liminf_{r \rightarrow 0} \frac{\log \tilde{\nu}_q(B(x, r))}{\log r},$$

$$\operatorname{Dim} E \geq \operatorname{ess\,sup}_{x \in E, \tilde{\nu}_q} \limsup_{r \rightarrow 0} \frac{\log \tilde{\nu}_q(B(x, r))}{\log r}.$$

However, using the same method as Lemma 11, the local Hölder exponent of  $\tilde{\nu}_q$  can be computed by applying general balls as well as dyadic intervals. For  $\tilde{\nu}_q$ -almost every  $x \in [0, 1]$ ,

$$\liminf_{r \rightarrow 0} \frac{\log \tilde{\nu}_q(B(x, r))}{\log r} = \liminf_{n \rightarrow \infty} \frac{\log \tilde{\nu}_q(I_n(x))}{\log 2^{-n}},$$

$$\limsup_{r \rightarrow 0} \frac{\log \tilde{\nu}_q(B(x, r))}{\log r} = \limsup_{n \rightarrow \infty} \frac{\log \tilde{\nu}_q(I_n(x))}{\log 2^{-n}}.$$

Since dyadic intervals correspond to cylinders, we refer to [4, 7] and present the proof of Theorem 5.

*Proof of Theorem 5.* The strong law of large numbers shows that

$$\liminf_{n \rightarrow \infty} \frac{\log_2 \tilde{\nu}_q(I_n(x))}{-n} = \min\{h(\tilde{a}), h(\tilde{b})\},$$

$$\limsup_{n \rightarrow \infty} \frac{\log_2 \tilde{\nu}_q(I_n(x))}{-n} = \max\{h(\tilde{a}), h(\tilde{b})\},$$

for  $\tilde{\nu}_q$ -almost every  $x$ , where

$$h(\tilde{a}) = - \sum_{i=1}^2 \tilde{a}_i \log_2 \tilde{a}_i \quad \text{and} \quad h(\tilde{b}) = - \sum_{i=1}^2 \tilde{b}_i \log_2 \tilde{b}_i.$$

So it deduces from Lemma 12 that

$$\dim X_\nu(\alpha) \geq \min\{h(\tilde{a}), h(\tilde{b})\},$$

$$\text{Dim } X_\nu(\alpha) \geq \max\{h(\tilde{a}), h(\tilde{b})\}.$$

And these two inequalities remain valid if we replace  $\nu$  with  $\mu$ .

At the same time, one obtains

$$\begin{aligned} h(\tilde{a}) &= \theta_a(q_a) - q_a \theta'_a(q_a) = \theta_a^*(-\theta'_a(q_a)) = \theta_a^*(\alpha), \\ h(\tilde{b}) &= \theta_b(q_b) - q_b \theta'_b(q_b) = \theta_b^*(-\theta'_b(q_b)) = \theta_b^*(\alpha). \end{aligned}$$

Recall that the upper bounds of the dimensions of the level sets have been given by Olsen [5]. So for any  $\alpha \in (-\log s_1, -\log s_2)$ , we have

$$\dim X_\mu(\alpha) = \dim X_\nu(\alpha) = b_\mu^*(\alpha) = b_\nu^*(\alpha).$$

And for  $\alpha \in (-\log s_1, -\log s_2)$  such that

$$\max\{\theta_a^*(\alpha), \theta_b^*(\alpha)\} = B_\mu^*(\alpha),$$

we have

$$\text{Dim } X_\mu(\alpha) = \text{Dim } X_\nu(\alpha) = B_\mu^*(\alpha) = B_\nu^*(\alpha).$$

□

**Corollary 13.** *For any  $q$  such that  $B'_\mu(q)$  exists, denote  $\alpha = -B'_\mu(q)$ . If  $\alpha \in (-\log s_1, -\log s_2)$ , then we have*

$$\dim X_\mu(\alpha) = \dim X_\nu(\alpha) = b_\mu^*(\alpha) = b_\nu^*(\alpha),$$

$$\text{Dim } X_\mu(\alpha) = \text{Dim } X_\nu(\alpha) = B_\mu^*(\alpha) = B_\nu^*(\alpha).$$

*Proof.* It is easy to see that when  $B'_\mu(q)$  exists,  $B'_\mu(q)$  coincides with either  $\theta'_a(q)$  or  $\theta'_b(q)$ . Without loss of generality, we may assume that for such  $q$ ,  $B'_\mu(q) = \theta'_a(q)$ , then  $B_\mu(q) = \theta_a(q)$ . So

$$\theta_a^*(-\theta'_a(q)) = \theta_a(q) - q\theta'_a(q) = B_\mu(q) - qB'_\mu(q) = B_\mu^*(-B'_\mu(q)),$$

which implies

$$\theta_a^*(\alpha) = B_\mu^*(\alpha).$$

But this means

$$\max\{\theta_a^*(\alpha), \theta_b^*(\alpha)\} = B_\mu^*(\alpha).$$

□

*Remark 14.* One can before projection compose with an isometry of the symbolic space in Lemma 6, and thus in Theorem 4 and Theorem 5. To be precise, let  $g : (\partial\mathcal{A}^*, d) \rightarrow (\partial\mathcal{A}^*, d)$  be an isometry, and denote by  $\nu_g$  the image measure of  $\mu$  under  $\gamma \circ g$ . Then it is easy to see that for any two words  $x$  and  $y$ ,  $g(x \wedge y) = g(x) \wedge g(y)$ . So the proof of Inequality (4) is valid if we replace the measure  $\nu$  with  $\nu_g$ .

Of course, Gray codes are isometries. As seen in [3, 4, 7], if  $g$  is a Gray code, then the measure  $\nu_g$  becomes a doubling measure on  $[0, 1]$ . But for general  $g$ ,  $\nu_g$  needs not be doubling.

### Acknowledgements

The author is grateful to Professor Jacques Peyrière for his patient guidance and helpful comments. The author would also like to thank Mr. Zhihui Yuan for the beneficial discussions the author had with him.

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